Hodge Theory Lecture 7

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I Will note that the title of this is somewhat of a misnomer, as the Dirichlet problem requires functions or forms to be 0 on the boundary, while we don't have a boundary.

The notes this week will be fairly short, as this follows in large part from my notes.

Lemma 0.1. *A sequence* $\{x_n\} \subseteq H^1$ *(or* $H^1\Omega^k(M)$*), converges weakly to* x *iff* $f(x_n) \to f(x)$ *for all bounded linear functionals* f

Proof. Riesz Representation Theorem $\square$

Lemma 0.2. *Given* $S \subseteq H^1$ *(or* $H^1\Omega^k(M)$*) the set* $\{v \in H^1 : \langle v, w\rangle_{L^2} = 0, \forall w \in S\}$ *is weakly closed.*

Proof. We have for each $w \in S$ the set $w^\perp = \{v \in H^1 : \langle v, w\rangle_{L^2} = 0\}$ is weakly closed, since $v \mapsto \langle v, w\rangle_{L^2}$ is a bounded linear functional.

Therefore,

$$\{v \in H^1 : \langle v, w\rangle_{L^2} = 0, \forall w \in S\} = \bigcap_{w \in S} w^\perp$$

is weakly closed $\square$

Lemma 0.3. *The functional* $w \mapsto \|dw\|_{L^2}^2 + \|d^*w\|_{L^2}^2$ *is weakly sequentially lower-semicontinuous*

Proof. Every weakly convergent sequence is bounded in norm, hence by Rellich-Kondrachov it is convergent in the L^2 norm
Thus we have that, since norms are weakly sequentially lower-semicontinuous and

$$\sqrt{\|(d+d^*)w\|_{L^2}^2 + \|w\|_{L^2}^2}$$

is an equivalent norm on Sobolev differential forms,

$$\liminf(\|dw_n\|_{L^2}^2 + \|d^*w_n\|_{L^2}^2) \geq \|dw\|_{L^2}^2 + \|d^*w\|_{L^2}^2$$

since d, d^* have orthogonal images. □

Lemma 0.4. *The set*

$$\{v \in H^1 : \|v\|_{L^2} = 1, \langle v, w\rangle_{L^2} = 0, \forall w \in \ker \triangle_k\} = \{v \in H^1 : \|v\|_{L^2} = 1, v \perp_{L^2} \ker \triangle_k\}$$

is weakly closed

Proof. The set $\{v \in H^1 : \|v\|_{L^2} = 1\}$ is clearly weakly closed, hence the set above is the intersection of closed sets □

Theorem 0.1. *The minimum on* $H^1\Omega^k(M)$ *of*

$$\min_{v \neq 0, v \perp_{L^2} \ker \triangle_k} \frac{B[v,v]}{\|v\|_{L^2}^2} = C_{Poin} > 0$$

where $B[u,v] = \langle dv, dw\rangle_{L^2} + \langle d^*v, d^*w\rangle_{L^2}$

Proof. Note that by scaling $v \mapsto \frac{v}{\|v\|_{L^2}}$ this is equivalent to showing that

$$\min_{\|v\|_{L^2}=1, v \perp_{L^2} \ker \triangle_k} B[v,v] = c > 0$$

but this minimum is achieved, and clearly we have that, by taking the derivative of $\frac{B[v,v]}{\|v\|_{L^2}^2}$, we find that

$$B[u,v] - c\langle u,v\rangle_{L^2} = 0, \forall u \in \perp_{L^2} \ker \triangle_k$$

and in particular $c = 0$ implies that

$$B[u,v] = 0 \Rightarrow B[u,u] = 0 \Rightarrow u \in \ker \triangle_k$$

which is impossible. □

Theorem 0.2. *Given $f \in L^2\Omega^k(M)$ with $f \perp_{L^2} \ker \triangle_k$, there exists a solution $g \in H^2\Omega^k(M)$ of $\triangle_k g = f$ with $g \perp_{L^2} \ker \triangle_k$*

Proof. We minimize the functional

$$B[u,u] - \langle u,f\rangle_{L^2}$$

over $u \in H^1\Omega^k(M)$ with $u \perp_{L^2} \ker \triangle_k$
Note that given $\varepsilon > 0$ there is some $C(\varepsilon) > 0$ so that

$$\langle u,f\rangle_{L^2} \leq \|u\|_{L^2} \|g\|_{L^2} \leq \varepsilon \|u\|_{L^2} + C(\varepsilon) \|g\|_{L^2}$$

Also, by the prior theorem

$$B[u,u] \geq \frac{1}{2}B[u,u] + \frac{C_{Poin}}{2} \|v\|_{L^2}^2$$

and hence if $\frac{C_{Poin}}{4} = \varepsilon > 0$ we find that

$$B[u,u] \geq \min\{\frac{C_{Poin}}{4}, \frac{1}{2}\} \|u\|_{H^1}^2 - C(\varepsilon) \|g\|_{L^2}$$

This is clearly coercive, and also evidently the functional is weakly sequentially lower-semicontinuous. Thus g is a weak solution. By regularity theory we have the desired result. □

I will note that this solution is unique, which can be seen by taking the second derivative, which is strictly positive along every line.

Theorem 0.3. *Every $\omega \in \Omega^k(M)$ can be written as $\omega_H + d\alpha + d^*\beta$ where ω_H is harmonic. This sum is mutually orthogonal in that every two of these are orthogonal.*

Proof. Combine the last theorem, plus extra regularity since ω is smooth, with $(\omega - P_H\omega) \perp_{L^2} \ker \triangle_k$, where P_H is the orthogonal projections onto the harmonic forms. We know this exists since the set of harmonic forms is finite dimensional. □

Corollary 0.1. *Every de Rham cohomology class has a harmonic representative*

Proof. If $d\omega = 0$ then

$$0 = \langle dd^*\beta, \beta \rangle_{L^2} = \langle d^*\beta, d^*\beta \rangle_{L^2}$$

and hence

$$\omega = \omega_H + d\alpha$$

and thus ω_H is the form we are looking for. □